

# Generalized particle dynamics: modifying the motion of particles and branes

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## Abstract

We construct a generalized dynamics for particles moving in a symmetric space-time, i.e. a space-time admitting one or more Killing vectors. The generalization implies that the effective mass of particles becomes dynamical. We apply this generalized dynamics to the motion of test particles in a static, spherically symmetric metric. A significant consequence of the new framework is to generate an effective negative pressure on a cosmological surface whose expansion is manifest by the particle trajectory via embedding geometry [5, 7, 15, 16]. This formalism thus may give rise to a source for dark energy in modelling the late accelerating universe.

## 1 Introduction

According to Einstein's theory of general relativity (GR) the motion of test particles in a specific background geometry is described by geodesics of space-time. However, for particles with spin and/or charge deviations are expected due to spin-orbit coupling [1, 2] or external fields.

Tests of Einstein's theory are usually limited to special geometries, such as that of spherical masses (e.g., stars). Some indications exist that deviations

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from geodesic motion arises over long distances and time scales, such as for example the famous Pioneer anomaly [3].

In this paper we propose a modification of Einstein's equations of motion for particles in a fixed symmetric space-time, i.e. a space-time admitting one or more Killing vectors. The main ingredient is a coupling between the constant of motion associated with a Killing vector and the orbit of the particle, generalizing the concept of spin-orbit coupling. Like the equation of motion for particles in flat Minkowski space, the action is only invariant under special co-ordinate transformations defined by the Killing vectors [4]. In the context of GR we therefore expect our prescription to arise in specific geometries as an effective action, taking into account special effects from the background geometry.

We apply our formalism to the specific case of Schwarzschild geometry. We introduce an additional non-minimal coupling between orbital motion and angular momentum and we make explicit the changes in the effective potential for test particles arising from our modification. We show how the conventional results are recovered in the limit of vanishing coupling.

An interesting cosmological application arises from applying our new dynamics to brane models of cosmology. Observing that new physics seems to be required to explain dark energy as the dominant constituent of our present-day universe, it would be very interesting as well as convenient if one can understand the negative pressure behavior - a signature of dark energy - from a modified brane dynamics. Indeed, the modifications of dynamics we propose can incorporate this feature in an effective theory, by *non-minimal* coupling to conventional gravity.

Taking cue from the braneworld gravity [5] which arises from the embedding of the 4-dimensional world in higher dimension, we follow closely the thin-shell formalism of [6, 7] where one develops the Friedman equations for cosmology, governed by the induced gravity on the brane. In our case we use the effective metric emerging from the particle back reaction. We show that *with this extension one can have a universe which interpolates between the early decelerating and late time accelerating phases*. Indeed, it is satisfying that in the present setup one can visualize the early decelerating and late time accelerating phases of the universe in a unified framework.

In our analysis of the model we exploit both lagrangian and the hamiltonian framework, using Dirac's formulation [8] of constrained dynamics. Finally the relevance of Schwarzschild geometry to a wide class of sources, as implied by Birkhoff's theorem [9], motivates our choice of studying this

example in more detail.

## 2 The generalized particle formalism

We propose a generalized action for point particles in a space-time admitting one or more Killing vectors, taking the form

$$S = \int L d\tau = m \int d\tau \left[ \frac{1}{2e} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \frac{e}{2} - \lambda g_{\mu\nu} \xi^\mu \dot{x}^\nu + \frac{e\lambda^2}{2} g_{\mu\nu} \xi^\mu \xi^\nu + \frac{e\beta\lambda^2}{2} \right]. \quad (1)$$

Here  $\tau$  is the worldline evolution parameter,  $x^\mu(\tau)$  are the particle co-ordinates,  $e(\tau)$  is the worldline einbein introduced to make the action reparametrization invariant [10], and  $\lambda(\tau)$  is an auxiliary worldline scalar variable. Furthermore  $\beta$  is a constant, whilst the metric  $g_{\mu\nu}(x)$  and the vector  $\xi^\mu(x)$  are functions of the co-ordinates  $x^\mu$ .

The action is invariant under infinitesimal co-ordinate transformations of the form  $\delta x^\mu = \alpha \xi^\mu(x)$ , provided the Lie-derivative of the metric with respect to  $\xi$  vanishes:

$$\xi^\lambda \partial_\lambda g_{\mu\nu} + g_{\nu\lambda} \partial_\mu \xi^\lambda + g_{\mu\lambda} \partial_\nu \xi^\lambda = 0. \quad (2)$$

This shows that  $\xi^\mu(x)$  is the Killing vector associated with the symmetry of the metric. If the metric admits more than one Killing vector, the action (1) can be extended accordingly.

In the following we use the notation,

$$\dot{x}^2 = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu, \quad \xi \cdot \dot{x} = g_{\mu\nu} \xi^\mu \dot{x}^\nu, \quad \xi^2 = g_{\mu\nu} \xi^\mu \xi^\nu. \quad (3)$$

The lagrangian equations of motion for the worldline variables  $(e, \lambda)$  then imply

$$\frac{\dot{x}^2}{e^2} = -1 + \lambda^2 (\xi^2 + \beta) \quad ; \quad \lambda = \frac{1}{e} \frac{\xi \cdot \dot{x}}{\xi^2 + \beta}. \quad (4)$$

These equations can be used to eliminate  $e$  and  $\lambda$  from the action, leading to a classically equivalent expression

$$\tilde{S} = -m \int d\tau \sqrt{-\dot{x}^2 + \frac{(\xi \cdot \dot{x})^2}{\xi^2 + \beta}}. \quad (5)$$

It is interesting to note that Eq (2.5) is a somewhat generalized version of the point particle action derived in [11] in the context of noncommutative

Snyder geometry [12]. The associated canonical momenta are

$$p_\mu = \frac{\partial \tilde{S}}{\partial \dot{x}^\mu} = \frac{m}{e} g_{\mu\nu} \left( \dot{x}^\nu - \xi^\nu \frac{\xi \cdot \dot{x}}{\xi^2 + \beta} \right), \quad (6)$$

which satisfy the constraints

$$\xi \cdot p = m\beta\lambda, \quad p^2 + \frac{1}{\beta}(\xi \cdot p)^2 + m^2 = 0. \quad (7)$$

In these last equations  $e$  and  $\lambda$  are to be interpreted as short-hand notation for the solutions of eqs. (4).

#### *Hamiltonian formulation*

To analyze the dynamics implied by the action (1), we follow the hamiltonian analysis of constrained systems as formulated by Dirac [8]. The canonical momenta are given by

$$p_e = \frac{\partial L}{\partial \dot{e}} = 0 \quad ; \quad p_\lambda = \frac{\partial L}{\partial \dot{\lambda}} = 0 \quad ; \quad p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = mg_{\mu\nu} \left( \frac{1}{e} \dot{x}^\nu - \lambda \xi^\nu \right). \quad (8)$$

The Hamiltonian follows,

$$H = p_e \dot{e} + p_\lambda \dot{\lambda} + p_\mu \dot{x}^\mu - L = \frac{e}{2m} [p^2 + m^2 + 2m\lambda(\xi \cdot p) - \beta m^2 \lambda^2]. \quad (9)$$

We have two primary constraints

$$\psi_1 \equiv p_e \approx 0 \quad ; \quad \psi_2 \equiv p_\lambda \approx 0, \quad (10)$$

leading to two secondary constraints,

$$\dot{\psi}_2 \equiv \psi_3 \equiv e(\xi \cdot p - m\beta\lambda) \approx 0; \quad \dot{\psi}_1 \equiv \psi_4 \equiv p^2 + m^2 + \frac{1}{\beta}(\xi \cdot p)^2 \approx 0. \quad (11)$$

Notice that  $e$  is not allowed to vanish and hence  $\psi_3$  can be replaced by  $\tilde{\psi}_3 \equiv \xi \cdot p - m\lambda\beta \approx 0$ . Equivalently, we can remove the auxiliary pair  $e, p_e$  by fixing the gauge  $e = 1$ . The remaining three constraints are non-commuting in general and we may obtain the first-class constraint (related to the generator of the reparametrization invariance) [8] by appropriate linear combination of  $\psi_2, \psi_3, \psi_4$ . However, as  $\xi^\mu$  is a Killing vector one can check that  $\psi_4$  commutes with itself as well as with  $\psi_2, \psi_3$ . Hence  $\psi_4$  is the first-class

constraint. Indeed,  $\psi_4$  coincides with the hamiltonian (9) and the lagrangian constraint (7) upon enforcing the remaining second-class constraints  $\psi_2$  and  $\tilde{\psi}_3$ . Thus

$$2mH = \psi_4 = p^2 + m^2 + \frac{1}{\beta}(\xi \cdot p)^2 = 0. \quad (12)$$

Clearly this constraint is a generalization of the normal mass-shell condition  $p^2 + m^2 \approx 0$  one is familiar with.

Of course, we must also deal with the second-class constraints by replacing the canonical Poisson brackets by Dirac brackets [8], but in the present case this does not affect the canonical symplectic structure of the remaining variables. As in the conventional case our gauge choice implies  $x^0 = t$  and we can proceed to the standard hamiltonian dynamics.

### 3 An application to Schwarzschild geometry

We now apply our generalized dynamics to the case of a particle in a Schwarzschild background, with line element (in natural units  $c = G = 1$ )

$$g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \quad (13)$$

As is well-known, the symmetries of the metric lead to four Killing vectors: one related to time translation (conservation of energy) and the other three deal with rotational symmetry, i.e. conservation of angular momentum. Out of these three, one deals with the magnitude whereas the other two are related to the direction of angular momentum, which means that the particle will move in a plane. Let us choose this plane to be the equatorial plane  $z = 0$ , or  $\theta = \frac{\pi}{2}$ . Then we can reduce the above metric on the equatorial plane to

$$g_{mn}dx^m dx^n = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2d\phi^2 \quad (14)$$

The Killing vectors associated with time translations and with rotations in the equatorial plane carry over to the reduced metric. In particular, in this frame the Killing vector associated with angular momentum is  $\xi_{(\phi)}^\mu = (0, 0, 1)$ . Incorporating this Killing vector into our general action (1), we have

$$S = \int L d\tau = m \int \left[ \frac{1}{2e} \left( -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1 - \frac{2M}{r}} + r^2\dot{\phi}^2 \right) - \frac{e}{2} - \lambda r^2\dot{\phi} \right]$$

$$+ \frac{e\lambda^2}{2} (r^2 + \beta) \Big] d\tau. \quad (15)$$

The canonical momenta turn out to be

$$\begin{aligned} p_e &= \frac{\partial L}{\partial \dot{e}} = 0 \quad ; \quad p_\lambda = \frac{\partial L}{\partial \dot{\lambda}} = 0 \quad , \\ p_r &= \frac{\partial L}{\partial \dot{r}} = \frac{m\dot{r}}{e(1 - \frac{2M}{r})} \quad ; \quad p_t = \frac{\partial L}{\partial \dot{t}} = -\frac{m\dot{t}}{e} \left(1 - \frac{2M}{r}\right) \quad , \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = \frac{m}{e} r^2 \dot{\phi} - m\lambda r^2. \end{aligned} \quad (16)$$

The Hamiltonian is

$$\begin{aligned} H &= p_e \dot{e} + p_\lambda \dot{\lambda} + p_r \dot{r} + p_t \dot{t} + p_\phi \dot{\phi} - L \\ &= \frac{e}{2m} \left[ m^2 - \frac{p_t^2}{1 - \frac{2M}{r}} + \left(1 - \frac{2M}{r}\right) p_r^2 + \frac{p_\phi^2}{r^2} + 2m\lambda p_\phi - \beta m^2 \lambda^2 \right] \end{aligned} \quad (17)$$

Performing the same constraint analysis as before whilst fixing the gauge  $e = 1$ , we find the modified mass-shell constraint and the relation,

$$2mH = m^2 - \frac{p_t^2}{1 - \frac{2M}{r}} + \left(1 - \frac{2M}{r}\right) p_r^2 + \left(\frac{1}{r^2} + \frac{1}{\beta}\right) p_\phi^2 = 0 \quad ; \quad p_\phi = m\beta\lambda. \quad (18)$$

Consequently the Hamiltonian equations of motion give the two conserved quantities,

$$\dot{p}_\phi = \{p_\phi, H\} = 0 \quad , \quad (19)$$

$$\dot{p}_t = \{p_t, H\} = 0. \quad (20)$$

So we have  $p_\phi = l$  and  $p_t = E$ , where  $l$  and  $E$  are two different constants. It is worth mentioning here that  $l$  and  $E$  are nothing but the angular momentum and energy respectively, for the particle moving in Schwarzschild background, but now depending on the parameter  $\beta$  as well. In the large  $\beta$  limit the standard results are regained.

This is the Hamiltonian way of reproducing the property that for any Killing vector  $\xi^\mu$

$$\xi \cdot p = c, \quad (21)$$

where  $c$  is time independent. For the Killing vector shown before,

$$\xi_{(\phi)} \cdot p = p_\phi = l = m\beta\lambda, \quad (22)$$

where  $l$  is a constant and the last equality follows from the constraint. From the time-like Killing vector:  $\xi_{(t)}^\mu = (1, 0, 0)$  we obtain energy, the other constant of motion,

$$\xi_{(t)} \cdot p = p_t = E. \quad (23)$$

These are just the same quantities we have obtained previously by using the Dirac brackets. Thus, those constants are nothing but the angular momentum and energy in the generalized particle dynamics, respectively.

## 4 Analysis and interpretation of the Effective Potential

By substitution of the radial momentum (16) and the constants of motion (23) and (22) into the mass-shell condition (18) we find an expression for the radial velocity:

$$\dot{r}^2 + \frac{2}{m} V_{eff}(r) = \varepsilon^2, \quad (24)$$

with  $\varepsilon = E/m$  and

$$2mV_{eff} = \left(1 - \frac{2M}{r}\right) \left(m^2 + \frac{l^2}{\beta} + \frac{l^2}{r^2}\right). \quad (25)$$

In the limit  $\beta \rightarrow \infty$  this reduces to the standard effective potential for particles in a Schwarzschild background.

Following standard practice, we shall now analyze the effective potential in order to extract information about the particle dynamics. Introducing an  $l$ -dependent effective mass

$$\tilde{m}^2 = m^2 + \frac{l^2}{\beta}, \quad (26)$$

we can rewrite  $V_{eff}$  in the form,

$$2mV_{eff} = \left(1 - \frac{2M}{r}\right) \left[\tilde{m}^2 + \frac{l^2}{r^2}\right]. \quad (27)$$

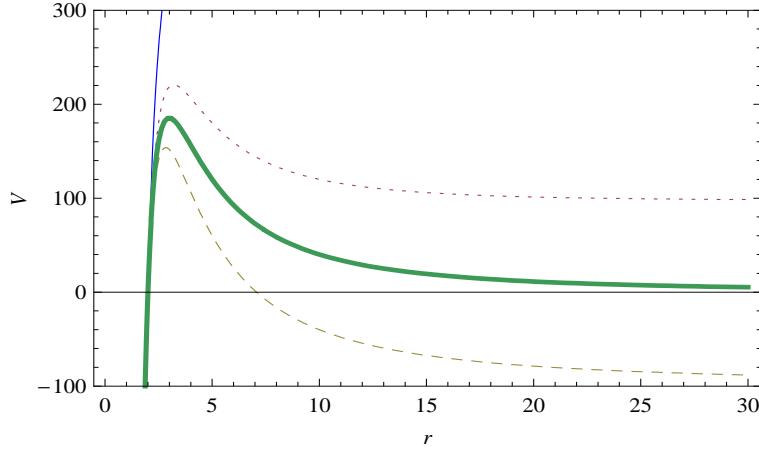


Figure 1: Plot for the variation of the effective potential with  $r$  for  $l = 100$  with different values of  $\beta$

The effect of different values of  $\beta$  on the dynamics of a particle with mass  $m$  can be seen in Figure 1, where we plot  $V(r)$  vs.  $r$  for different  $\beta$  at a fixed  $l = 100$ . In this figure the normal line corresponds to  $\beta = 10$  whereas the dotted, thick and dashed lines represent the values  $\beta = 50, 10^5, -50$  respectively. Also, we have taken  $M = 1$ ,  $m = 0.1$ . We now discuss the various cases in some more detail.

### **Positive $\beta$ :**

The  $\beta \rightarrow \infty$  limit, where the conventional particle is recovered, is given by the  $\beta = 10^5$  plot. On the other hand, the potential diverges for a small value of  $\beta$  (indicated by normal line) so that one cannot have a bound state for a particle in the small  $\beta$  limit. This happens due to a huge amount of the binding energy required from the additional amount of mass term given by  $l^2/\beta$ . In order to have a stable orbit for the particle, the effective potential  $V(r)$  should have a minimum; so we must have

$$\frac{dV}{dr} = 0. \quad (28)$$

The above relation gives us a quadratic equation to solve and the equation is the following:

$$M\tilde{m}^2r^2 - l^2r + 3Ml^2 = 0. \quad (29)$$

The two solutions of the above equation are given by:

$$r_{\pm} = \frac{l^2 \pm \sqrt{l^4 - 12M^2l^2(m^2 + \frac{l^2}{\beta})}}{2M(m^2 + \frac{l^2}{\beta})}. \quad (30)$$

One of these two solutions represents a maximum, i.e. an unstable orbit and the other represents a minimum, i.e. a stable orbit. The stability condition turns out to be,

$$l^2 \geq \frac{12M^2m^2\beta}{\beta - 12M^2}. \quad (31)$$

**Negative  $\beta$ :**

However, strikingly new behavior appears for negative  $\beta$  with  $l^2 < |\beta|m^2$ . This is seen in the Figure 1 for  $\beta = -50$ . The effective potential changes sign after some value of  $r$ , indicating a change in the behavior of the particle trajectory at some point. The cosmological implications of this sign-flip will be discussed in the next section.

## 5 Cosmological implications of negative $\beta$

So far we have analyzed the significance of the dynamics of the generalized particle. In this section we shall point out a crucial implication of the scenario in the cosmological context. From embedding geometry using the Gauss-Codazzi equation [14] it can be shown that a  $(D-1)$ -dimensional surface representing a cosmological Friedmann-Robertson-Walker (FRW) metric can be embedded consistently in a  $D$ -dimensional black hole space-time in such a way that the expansion of the FRW surface is realized by the particle trajectory along the radial direction in the gravitational field of the black hole [5, 6, 7, 15, 16]. In the present article, the particle motion represents a 3-dimensional FRW metric and the effective potential contains essential information for the evolution of the embedded cosmological surface. Below we shall show how this can be realized, followed by a discussion of its cosmological implications.

Let us take a careful look at the expression for the effective potential (27). It contains additional terms arising from  $\beta$  in the generalized particle approach. It is straightforward to verify that the scenario is identical to the

motion of a point mass in the field of a static, spherically symmetric metric

$$ds_4^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2d\Omega_2^2 \quad (32)$$

where

$$d\Omega_2^2 = \frac{d\sigma^2}{1 - k\sigma^2} + \sigma^2 d\phi^2, \quad (33)$$

provided the metric function  $F(r)$  is chosen as

$$F(r) = \left(1 - \frac{2M}{r}\right) \left(\frac{\tilde{m}^2 r^2 + l^2}{m^2 r^2 + l^2}\right) \quad (34)$$

The geodesic equations associated with this metric (32) reproduce precisely the same effective potential as that in eq. (27). How the above form of the metric can be derived directly from solving the Einstein equations with some specific source is presently an open problem. The reader may however note that this effective metric falls in the class of several complicated spherically symmetric metrics available in the literature [17] (Some more exotic metrics are listed in [18]); here the only point is that this effective metric arises from the generalized particle dynamics. In reality, of course, the background space-time still remains of the Schwarzschild type.

Our intention is, rather, to embed a 3-dimensional FRW metric

$$ds_3^2 = -d\tau^2 + a^2(\tau)d\Omega_2^2 \quad (35)$$

(where  $d\Omega_2^2$  is the two-space representing the flat, open or closed spaces) into the space-time given by eq. (32). With the effective metric function (34), and identifying the scale factor with the radial trajectory so that the expansion of the cosmological universe is realized by the radial motion of the particle, the tangents (4-velocity) and normals to the surface satisfying the orthonormality and normalization conditions are given by

$$\begin{aligned} u^\mu &\equiv \left( \frac{\sqrt{F(a) + \dot{a}^2}}{F(a)}, \dot{a}, 0, 0 \right) \\ n^\mu &\equiv \left( -\frac{\dot{a}}{F(a)}, -\sqrt{F(a) + \dot{a}^2}, 0, 0 \right) \end{aligned} \quad (36)$$

Here, and throughout the rest of the discussion, we identify  $r(\tau)$  with the scale factor  $a(\tau)$ .

Further, the extrinsic curvature turns out to be

$$K_{ij} = \frac{\sqrt{F(a) + \dot{a}^2}}{a} \tilde{g}_{ij} ; \quad K_{\tau\tau} = \frac{d}{da} \left( \sqrt{F(a) + \dot{a}^2} \right) \quad (37)$$

where  $\tilde{g}_{\mu\nu}$  is the induced metric of the 3-dimensional FRW surface.

The junction conditions, along with  $Z_2$ -symmetry, relates the extrinsic curvature to the effective surface stress-energy tensor  $S_{\mu\nu}$  by

$$K_{\mu\nu} = -8\pi G \left( S_{\mu\nu} - \frac{1}{3} S \tilde{g}_{\mu\nu} \right) \quad (38)$$

With the extrinsic curvature (37), the square of the above equation, immediately leads to

$$\left( \frac{\dot{a}}{a} \right)^2 = -\frac{F(a)}{a^2} + \frac{8\pi G}{3} \rho \quad (39)$$

where  $\rho$  is the effective surface density that arises from matter and the surface tension. Written explicitly in terms of the metric function (34), we have the Friedmann equation on the cosmological surface

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} + \frac{2M}{a^3} - \frac{l^2}{\beta} \left( \frac{1 - 2M/a}{l^2 + m^2 a^2} \right) \quad (40)$$

The effective matter conservation equation on the surface holds good. Consequently, we arrive at the following Raychaudhuri equation for expansion

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p) - \frac{M}{a^3} + T(\beta), \quad (41)$$

where the  $\beta$ -dependent terms are

$$T(\beta) = -\frac{l^2}{\beta(l^2 + m^2 a^2)^2} \left[ l^2 - \frac{Ml^2}{a} + Mm^2 a \right]. \quad (42)$$

The evolution of the 3-dimensional cosmological universe is governed by the above three equations. Eq. (41) needs special attention in this regard. The  $M/a^3$  term is a radiation-like effect (note the cosmology is now 3-dimensional) from the Weyl tensor of the black hole and hence is negligible for late time evolution. This is analogous to the dark radiation in a braneworld context [5]. The terms  $T(\beta)$ , however, are not so trivial, and in fact these terms give rise to the most important physical conclusion.

Indeed, the above correction to the evolution equations (41) of the embedded cosmological surface is quite significant when the  $\beta$  parameter takes negative value. With a negative  $\beta$ , the term outside the square bracket is positive definite. Consequently, this correction term  $T(\beta)$  has a positive contribution to the expansion equation (41) for the relevant region  $a > 2M$  (the particle trajectory is outside the black hole horizon), thereby resulting in an effective negative pressure which becomes significant at late time. This term essentially leads to late time accelerating phase. This is clearly demonstrated in **Figure 1**. The model is thus capable of explaining a transition from the decelerating to accelerating phases of the universe with an effective negative pressure arising from a negative  $\beta$ . Thus the framework has the potentiality to provide a source for dark energy.

Our claim can further be established from the analysis of deceleration parameter which is one of the major observable quantities for the present universe. Considering the terms containing  $\beta$  as the driving force for the acceleration of the universe (which implies that this is the dominant contribution to cosmic density at late time) and neglecting the effect of the matter sector, the deceleration parameter turns out to be

$$q = -\frac{\ddot{a}/a}{(\dot{a}/a)^2} = -\frac{l^2}{l^2 + m^2 a^2} - \frac{M}{a - 2M}. \quad (43)$$

This clearly reveals that the deceleration parameter is negative for the cosmologically relevant region  $a > 2M$ , confirming an accelerated expansion at late time. This behavior is further transparent from **Figure 2** which shows the variation of the deceleration parameter  $q$  with Hubble parameter  $H = \dot{a}/a$  for the representative values of the constants.

The behavior for the scale factor can also be analyzed to some extent with the above considerations of  $\beta$ -dominance. In the large  $l$  and small  $m$  limit, an approximate solution for the Friedmann equations can be obtained as

$$a(t) \approx M + \frac{1}{2} \left[ e^{t/\sqrt{-\beta}} - M^2 e^{-t/\sqrt{-\beta}} \right] \quad (44)$$

As already mentioned,  $\beta$  is negative here. In the above equation, the first term is nothing but a scaling and with  $M = 1$  the second term reduces to  $\sinh(t/\sqrt{-\beta})$ , which interpolates between a decelerating phase (matter-dominated) at early time and an accelerated expansion at late time. Thus our model behaves pretty close to  $\Lambda$ CDM, with the inverse of the  $\beta$  parameter

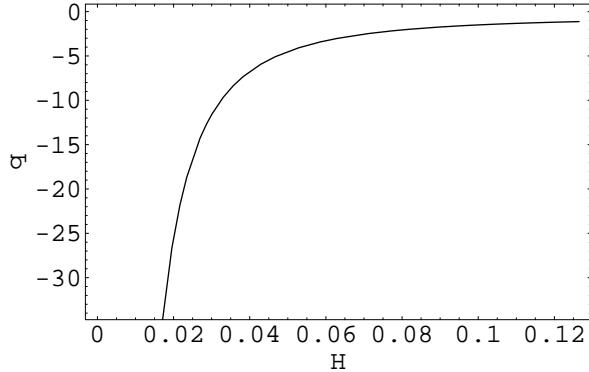


Figure 2: Variation of the deceleration parameter  $q$  with Hubble parameter  $H$  for  $l = 100$ ,  $\beta = -50$ ,  $M = 1$ ,  $m = 0.1$

playing the role of the cosmological constant. Because of this identification, it is worthwhile to emphasize that large negative  $\beta$  is consistent with small positive value for  $\Lambda$  that is favored observationally. This makes the scenario further interesting from observational ground since now one can calculate the other observable parameters and compare them with highly accurate observational data.

From the particle dynamics point of view, an effective negative pressure from a negative  $\beta$  is, indeed, what is expected. This will be transparent from the expression of the effective mass given in the last section by  $\tilde{m}^2 = m^2 + \frac{l^2}{\beta}$ . Clearly, a negative value for the  $\beta$  parameter will result in a reduced effective mass for the particle as compared to its physical mass ( $\tilde{m}^2 < m^2$ ). This apparent loss in mass will make the particle feel as if it is being acted upon by a repulsive force, which is concretized by an effective negative pressure in the Friedmann equations (40) and (41), thereby leading to a late-accelerating behavior for the cosmological surface from our analysis. The sign-flip in the effective potential in **Figure 1** for negative  $\beta$  shows this behavior.

This gives a major cosmological implication of the paradigm. One can readily see that the computations are literally the same for a 4-dimensional cosmological metric embedded in a 5-dimensional black hole space-time. Thus, there is absolutely no problem in applying the model to real-life cosmological analysis. A thorough investigation in this direction will be reported in future.

## 6 Summary and outlook

In this article we have developed a generalized framework for dynamics of a particle moving in a fixed background geometry, based on a reparametrization-invariant action containing the information of the symmetry properties of the space-time. In our analysis, we have considered hamiltonian as well as lagrangian formalisms and have shown that both of them lead to same results, thereby confirming the consistency of the theory.

Applying our framework to the case of Schwarzschild geometry, we have further analyzed the effective potential as usual in GR. Our analysis reveals some distinct features of the model. First, the physical mass of the particle is replaced by a more relevant effective mass which is a sum-total of the physical mass and an additional mass-like effect arising from the term  $\beta$  introduced in the theory. Secondly, using rigorous analysis with different plots for the variation of the effective potential with radial distance, we find that among a spectrum of possibilities for different value of the  $\beta$  parameter, the usual particle dynamics for Schwarzschild space-time is recovered only for the  $\beta \rightarrow \infty$  limit. Further, the effective mass results in an additional binding energy for the particle so that a stable orbit is now obtained for a larger value of the angular momentum. A last and significant outcome of our analysis is that we uncover a source for negative pressure analogous to dark energy for a cosmological metric moving along the particle trajectory.

Since throughout the effective potential analysis, the physical mass is replaced by the effective mass the quantitative estimations for a massive particle are going to be different from the GR counterparts. One can estimate the quantities and subject them to experimental verifications. Also, the action we choose is one of the simplest extension of the single-particle action that preserves reparametrization-invariance. More general actions can lead to more dramatic results. For example, a suitable choice of the action may lead to a noncommutative structure which may further be utilized to verify whether the space-time becomes noncommutative in the vicinity of the black hole horizon.

A thorough investigation of the evolution of the universe providing a late-accelerating phase with the modified Friedmann equations is one of the major open issues. At the very first point, it is interesting to look for any analytical or numerical solution for the scale factor from equations (40) and (41), which will explicitly show the late-accelerating behavior. Secondly, one can calculate different observable quantities [19], such as the deceleration parameter

$q$ , the Hubble parameter  $H(z)$ , age of the universe  $t_0$ , luminosity distance  $d_L$ , statefinder parameters  $\{r, s\}$  [20],  $Om(z)$  parameter, acceleration probe  $\bar{q}$  [21] and the other observable quantities as well. The next step is to confront them with observations. The next step is to confront them with observations. Some of the issues have been discussed to some extent in the present article. We hope to address some of these issues in near future.

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## References

- [1] A.J.Hanson, T.Regge and C.Teitelboim, *Constrained Hamiltonian System*, Roma, Accademia Nazionale Dei Lincei, (1976); S.Ghosh, Phys.Lett. B338 (1994) 235 (arXiv:hep-th/9406089); Erratum-ibid. B347 (1995) 468
- [2] R.H. Rietdijk and J.W. van Holten, Class. Quantum Grav. 7 (1990), 247; J.W. van Holten, *Relativistic Dynamics of Spin in Strong External Fields*, Proc. 4th Hellenic School on Elementary Particle Physics, Vol. II; Eds. E.N. Gazis, G. Koutsoumbas, N.D. Tracas and G. Zoupanos (Corfu, 1992)
- [3] S.G. Turyshev et al., Int. J. Mod. Phys. D15 (2006), 1
- [4] For studies on Killing vectors, directly relevant for the present work, see . J.W. van Holten, Phys.Rev.D75 (2007) 025027 (hep-th/0612216); R.H. Rietdijk and J.W. van Holten, Nucl.Phys.B472 (1996) 427 (hep-th/9511166); R.H. Rietdijk and J.W. van Holten, Nucl.Phys.B404 (1993) 42,(hep-th/9303112); Symmetries and motions in manifolds. J.W. van Holten and R.H. Rietdijk, J.Geom.Phys.11:559,1993 (hep-th/9205074).
- [5] R. Maartens, Living Rev. Relativity 7 (2004) 7.
- [6] M.Gogberashvili, Europhys.Lett. 49 (2000) 396-399 (arXiv:hep-ph/9812365v1).

- [7] C. Barcelo and M. Visser, Phys. Lett. B **482**, 183 (2000).
- [8] P. A. M. Dirac, *Lectures on Quantum Mechanics*, Yeshiva University Press, New York, 1964.
- [9] G. D. Birkhoff, *Relativity and Modern Physics*, Harvard University Press, Cambridge (1923). For a proof of generalised Birkhoff's theorem, see R. D'Inverno, *Introducing Einstein's Relativity*, Oxford University Press, Oxford (1995).
- [10] L. Brink. P. Di Vecchia and P. Howe, Nucl. Phys. B118 (1977), 76
- [11] R. Banerjee, S. Kulkarni and S. Samanta, JHEP **05** (2006) 077 (arXiv:hep-th/0602151).
- [12] H. S. Snyder, Phys. Rev. **71** (1947) 38.
- [13] S. M. Carroll, *Lecture Notes on General Relativity*, gr-qc/9712019
- [14] E. Poisson, *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics*, Cambridge University Press, Cambridge (2004); S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Clarendon Press, Oxford (1992).
- [15] J. Garriga and M. Sasaki, Phys. Rev. **D62** (2000) 043523; P. Kraus, JHEP **12**, 011 (1999); D. Ida, JHEP **09**, 014 (2000); S. Mukohyama, T. Shiromizu and K Maeda, Phys. Rev. D **62**, 024028 (2000); P. Bowcock, C. Charmousis and R. Gregory, Class. Quant. Grav. **17**, 4745 (2000).
- [16] S. Pal, Phys. Rev. D **74**, 024005 (2006); S. Pal, Phys. Rev. D **78**, 043517 (2008).
- [17] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Clarendon Press, Oxford (1992).
- [18] P. C. Vaidya, Nature **171** (1953) 260; C. W. Misner, Phys. Rev. **137** (1965) B1360; R. W. Lindquist, R. A. Schwartz and C. W. Misner, Phys. Rev. **137** (1965) B1364; V. V. Narlikar, G. K. Patwardhan and P. C. Vaidya, Proc. Natl. Inst. Sci. India **9**, 229 (1943); B. Kuchowicz, Acta Phys. Pol. **B 3**, 209 (1972); H. A. Buchdahl, Astrophys. J. **140**, 1512 (1964); A. Chamblin, S. W. Hawking and H. S. Reall, Phys. Rev. D **61**,

065007 (2000); C. Germani and R. Maartens, Phys. Rev. **D** **64** (2001) 124010; R. Casadio, A. Fabbri and L. Mazzacurati, Phys. Rev. **D** **65** (2002) 084040; P. Kanti and K. Tamvakis, Phys. Rev. **D** **65**, 084010 (2002); S. Shankaranarayanan and N. Dadhich, Int. J. Mod. Phys. **D** **13** (2004) 1095; A. N. Aliev and A. E. Gumrukcuoglu, Phys. Rev. **D** **71** (2005) 104027; A. S. Majumdar and N. Mukherjee, Int. J. Mod. Phys. **D** **14** (2005) 1095; S. Creek, R. Gregory, P. Kanti and B. Mistry, Class. Quant. Grav. **23**, 6633 (2006); A. L. Fitzpatrick, L. Randall and T. Wiseman, JHEP **0611**, 033 (2006); R. Gregory, Lect. Notes Phys. **769**, 259 (2009) [arXiv:0804.2595].

- [19] V. Sahni, Lect. Notes Phys. **653** (2004) 141; E. J. Copeland, M. Sami and S. Tsujikawa, Int. J. Mod. Phys. **D** **15**, 1753 (2006); J. Frieman, M. Turner and D. Huterer, arXiv:0803.0982 [astro-ph]
- [20] V. Sahni, T. D. Saini, A. A. Starobinsky and U. Alam, JETP Lett. **77** (2003) 201; Pisma Zh. Eksp. Teor. Fiz. **77** (2003) 249; U. Alam, V. Sahni, T. D. Saini and A. A. Starobinsky, Mon. Not. Roy. Astron. Soc. **344** (2003) 1057
- [21] V. Sahni, A. Shafieloo and A. A. Starobinsky, Phys. Rev. **D** **78**, 103502 (2008) [arXiv: 0807.3548]